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String Cosmological Solutions in Self-Creation Theory of Gravitation

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Abstract We have studied the problem of cosmic strings for Bianchi-I, II, VIII and IX string cosmological models in Barber's (Gen. Relativ. Gravit. **14**:117, 1982) second self—creation theory of gravitation. We have obtained some classes of solutions by considering different functional form for metric potentials. It is also observed that due to the presence of scalar field, the power index '*m*' of the metric coefficients has a range of values.

Keywords Strings · Cosmology · Scalar-tensor theories

1 Introduction

Barber (1982) proposed two 'self-creation' theories based on two sets of general relativistic field equations involving mater and scalar field. The first theory is a generalization, in some sense, of the Brans–Dicke [3] equations. The second theory is a modification of general relativity to a variable G-theory. Like the first theory, this can not be derived from an action principle. In his second self-creation theory, the gravitational coupling of the Einstein field equations is allowed to be a variable scalar on the space-time manifold.

It is postulated that this scalar couples to the trace of the energy momentum tensor. Hence the field equations, in Barber's second theory, are

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi}{\phi} T_{ij} \tag{1.1}$$

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and

$$\Box\phi = \frac{8\pi}{3}\eta T. \tag{1.2}$$

In this theory the scalar field does not directly gravitate, but simply divides the matter tensor, acting as a reciprocal gravitational constant. For a detailed discussion of the self-creation theories of gravitation one may refer to the work of Barber [1].

In recent years there has been lot of interest in the study of cosmic strings. Cosmic strings have received considerable attention as they are believed to have served in the structure formation in the early stages of the universe. Cosmic strings may be created during phase transitions in the early era [4] and they act as a source of gravitational field [7]. It is also believed that strings may be one of the sources of density perturbations that are required for the formation of large scale structures of the universe.

Letelier [7], Krori et al. [5, 6], Maharaj and Beesham [9], Raj and Shuchi [11], Bhattacharjee and Baruah [2], Mahanta and Mukharjee [8], Reddy [12], and Pant and Oli [10] are some of the authors who have studied various aspects of string cosmologies in general relativistic theory as well as in alternative theories of gravitation. Very recently Venkateswarlu and Pavan Kumar [14] have studied string cosmologies with extra dimensions in alternative theories of gravitation.

In this paper we consider the Bianchi type-I, II, VIII and IX metrics in the self-creation cosmology in the context of cosmic strings. By making use of Letelier's form of energy-momentum tensor for a cloud of string, we present some classes of solutions in Barber's second self-creation theory of gravitation. We have also discussed some properties of solutions obtained in different scenarios.

2 Metric and Field Equations

2.1 Bianchi Type-I Metric

We consider the spatially homogeneous and anisotropic Bianchi type-I space time as

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{2.1}$$

where A, B and C, are functions of 't' only.

The energy momentum tensor T_{ij} for strings is given by

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j, \tag{2.2}$$

$$u^i u_i = -\lambda^i x_i = 1 \quad \text{and} \quad u^i x_i = 0 \tag{2.3}$$

where ρ is the rest energy density of the cloud of strings with particles attached to them, λ is the tension density of the strings and $\rho = \rho_p + \lambda$, ρ_p being the rest energy of the particles. The velocity u^i describes the cloud 4-velocity and x^i represents the direction of anisotropy. For the weak, strong and dominant energy conditions, one finds that $\rho > 0$ and $\rho_p \geq 0$ and the sign of λ is unrestricted. We now consider x^i to be along x-axis so that $x^i = (0, A^{-1}, 0, 0)$.

With the help of (2.2) and (2.3), the field equations (1.1) and (1.2) for the metric (2.1) can be written explicitly as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = 8\pi\phi^{-1}\lambda, \tag{2.4}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=0, \tag{2.5}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}=0, \tag{2.6}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}=8\pi\phi^{-1}\rho, \tag{2.7}$$

$$\ddot{\phi}+\dot{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=-\frac{8}{3}\pi\eta(\lambda+\rho) \tag{2.8}$$

where the over head dot denotes derivative with respect to t.

From (2.5) and (2.6) it can be seen that $B=kC$, where k is an arbitrary constant. Therefore we have four equations in five unknowns. For deterministic solutions we need one assumption. We shall explore physically meaningful solutions of the field equations (2.4–2.8) by considering simplifying assumptions to the field variables A, B and C.

2.1.1 Classes of Solutions

Case 1: $B=C=t^m$, where m is an arbitrary constant. In this case (2.5) reduces to

$$\frac{\ddot{A}}{A}+\frac{\dot{A}m}{At}+\frac{m(m-1)}{t^2}=0 \tag{2.9}$$

which on integration yields

$$A=t^n \tag{2.10}$$

where $n=\frac{(1-m)\pm\sqrt{1+2m-3m^2}}{2}$. The value of n is real only if $1+2m-3m^2\geq 0$.

The rest energy density and tension density of strings are

$$8\pi\left(\frac{\lambda}{\phi}\right)=\frac{3m^2-2m}{t^2}, \tag{2.11}$$

$$8\pi\left(\frac{\rho}{\phi}\right)=\frac{2mn+m^2}{t^2}. \tag{2.12}$$

Case 1.1: If $m=\frac{2}{3}$ then $n=\frac{2}{3}$. In this case the solution can be expressed as

$$A=B=C=t^{\frac{2}{3}}, \tag{2.13}$$

$$8\pi\left(\frac{\lambda}{\phi}\right)=0, \tag{2.14}$$

$$8\pi\left(\frac{\rho}{\phi}\right)=\frac{4}{3t^2} \tag{2.15}$$

which corresponds to dust filled isotropic universe without strings and $\rho_p=\rho-\lambda=\frac{4}{3t^2}$.

Equation (2.8) in this case can be written as

$$\ddot{\phi}+\dot{\phi}\left(\frac{2}{t}\right)+\frac{4\eta}{9t^2}\phi=0$$

which on integration gives

$$\phi = C_1 t^{k_1} + C_2 t^{k_2} \tag{2.16}$$

where $k_1 = \frac{-1+\sqrt{1-\frac{16}{9}\eta}}{2}$ and $k_2 = \frac{-1-\sqrt{1-\frac{16}{9}\eta}}{2}$.

It is interesting to see that the above solution is similar to the solution obtained by Soleng [13] in Barber's second self-creation theory of gravitation when the source of gravitation field is a perfect fluid. It can be seen that if $\eta < \frac{9}{16}$, the scalar field is real and has singularity at $t = 0$ and $\phi \to 0$ as $t \to \infty$.

The cosmological parameters are given by

$$\text{Spatial volume} \quad V^3 = t^2,$$

$$\text{Scalar expansion} \quad \theta = \frac{2}{t},$$

$$\text{Deceleration parameter} \quad q = -3\theta^2\left[\dot{\theta} + \frac{1}{3}\theta^2\right] = \frac{4}{t^4},$$

$$\text{Shear scalar} \quad \sigma^2 = \frac{26}{9t^2}.$$

The deceleration parameter acts as an indicator of the existence of inflation. If $q > 0$ the model decelerates in its standard way while $q < 0$, the model inflates. Since $q > 0$, there is no inflation in the model in this case. Thus the model is isotropic and non-inflationary. The spatial volume becomes zero at $t = 0$ and this shows the expansion of the model with time. As $t \to \infty$, $V^3 \to \infty$, $\theta \to 0$, $A, B, C \to \infty$.

Case 1.2: When $m = \frac{1}{2}$ then $n = \frac{1+\sqrt{5}}{4}$ and the solution read

$$A = t^{\frac{1+\sqrt{5}}{4}}, \qquad B = C = t^{\frac{1}{2}}, \tag{2.17}$$

$$8\pi\left(\frac{\lambda}{\phi}\right) = -\frac{1}{4t^2}, \tag{2.18}$$

$$8\pi\left(\frac{\rho}{\phi}\right) = \frac{2+\sqrt{5}}{4t^2}. \tag{2.19}$$

It is observed that from (2.18) and (2.19), $\lambda < 0$, $\rho > 0$, $\rho_p = \rho - \lambda = \frac{1+\sqrt{5}}{4t^2} > 0$. Thus at early era strings exist with negative λ but then the particles exist with positive ρ_p and $\frac{\rho_p}{|\lambda|} > 1$. This implies that the energy conditions are satisfied.

Equation (2.8) can be expressed as

$$\ddot{\phi} + \left(\frac{5+\sqrt{5}}{4t}\right)\dot{\phi} + \frac{\eta(1+\sqrt{5})}{12t^2}\phi = 0 \tag{2.20}$$

whose solution is given by

$$\phi = C_1 t^{k_1} + C_2 t^{k_2} \tag{2.21}$$

where

$$k_1 = \frac{-(\frac{5+\sqrt{5}}{4}) + \sqrt{(\frac{5+\sqrt{5}}{4})^2 - (\frac{1+\sqrt{5}}{3})\eta}}{2}$$

and

$$k_2 = \frac{-(\frac{5+\sqrt{5}}{4}) - \sqrt{(\frac{5+\sqrt{5}}{4})^2 - (\frac{1+\sqrt{5}}{3})\eta}}{2}.$$

The cosmological parameters $V^3 = t^{\frac{5+\sqrt{5}}{4}}$, $\theta = \frac{5+\sqrt{5}}{4t}$, $q = \frac{225+125\sqrt{5}}{64t^4} > 0$ and $\sigma = \sqrt{\frac{251+97\sqrt{5}}{144t^2}}$. Since $q > 0$, the model decelerates in its standard way. It can be seen that $V^3 \to \infty, \theta \to 0$, $A \to \infty$, $B \to \infty$, and $C \to \infty$ as $t \to \infty$. It can also be observed that the scalar field is real and singular at $t = 0$.

Case 2: Suppose $B = C$ and $A = B^m$, then (2.5) reduces to

$$(m+1)\frac{\ddot{B}}{B} + m^2\frac{\dot{B}^2}{B^2} = 0. \tag{2.22}$$

Integrating the above equation we obtain

$$B = \left[\left(\frac{m^2+m+1}{m+1}\right)t\right]^{\frac{m+1}{m^2+m+1}}.$$

The solution of the field equations (2.4–2.8) can be written as

$$A = A_0 t^{\frac{m(m+1)}{m^2+m+1}}, \tag{2.23}$$

$$B = C = B_0 t^{\frac{m+1}{m^2+m+1}}, \tag{2.24}$$

$$8\pi\left(\frac{\lambda}{\phi}\right) = \frac{(1+2m)(1-m^2)}{(1+m+m^2)t^2}, \tag{2.25}$$

$$8\pi\left(\frac{\rho}{\phi}\right) = \frac{(m+1)^2(2m+1)}{(m^2+m+1)t^2} \tag{2.26}$$

where the arbitrary constant m satisfies $2m^3 + 3m^2 + m = 0$.

On equating λ and ρ, we get the above solution which is formally similar to the solution obtained by Pant and Oli [10] in general relativity.

Case 2.1: In case $m = 0$, then $B = C = B_0 t$, and $A = A_0$. Equation (2.8) in this case reduces to

$$\ddot{\phi} + \left(\frac{2}{t}\right)\dot{\phi} + \frac{2\eta}{3t^2}\phi = 0. \tag{2.27}$$

The solution of (2.27) can be expressed as

$$\phi = C_1 t^{k_1} + C_2 t^{k_2} \tag{2.28}$$

where

$$k_1 = \frac{-1 + \sqrt{1 - \frac{8}{3}\eta}}{2}, \qquad k_2 = \frac{-1 - \sqrt{1 - \frac{8}{3}\eta}}{2}$$

and

$$8\pi\left(\frac{\lambda}{\phi}\right) = \frac{1}{t^2} = 8\pi\left(\frac{\rho}{\phi}\right) \tag{2.29}$$

which describe Nambu (or geometric) strings in Barber's second self-creation theory. The cosmological parameters are $V^3 = t^2$, $\theta = \frac{2}{t}$, $q = \frac{8}{t^4}$ and $\sigma^2 = \frac{29}{9t^2}$.

Case 2.2: In case $m = -\frac{1}{2}$, the solution of the field equations (2.4–2.8) can be written as

$$A = A_0 t^{-\frac{1}{3}}, \qquad B = C = B_0 t^{\frac{2}{3}}, \tag{2.30}$$

$$8\pi\left(\frac{\lambda}{\phi}\right) = 8\pi\left(\frac{\rho}{\phi}\right) = 0. \tag{2.31}$$

It is interesting to note that the cosmic strings do not exist in this particular case. Also it can be seen that one of the field variables expands while the other contracts as $t \to \infty$.

From (2.8)

$$\ddot{\phi} + \frac{1}{t}\dot{\phi} = 0 \tag{2.32}$$

which on integration yields

$$\phi = a \log t + b \tag{2.33}$$

where a and b are integration constants.

The cosmological parameters are

$$\text{Spatial volume} \quad V^3 = t,$$

$$\text{Scalar expansion} \quad \theta = \frac{1}{t},$$

$$\text{Deceleration parameter} \quad q = -3\theta^2\left[\dot{\theta} + \frac{1}{3}\theta^2\right] = \frac{2}{t^4},$$

$$\text{Shear scalar} \quad \sigma^2 = \frac{19}{18t^2}.$$

The spatial volume at $V^3 = 0$ at $t = 0$ and $V^3 \to \infty$ as $t \to \infty$. Thus the model is expanding with time.

2.2 Bianchi Type-II, VIII and IX Metric

The line element for Bianchi type-II, VIII and IX can be expressed as

$$ds^2 = -dt^2 + R^2(dy^2 + f^2(y)dx^2) + S^2(dz + h(y)dx)^2 \tag{3.1}$$

where S and R are functions of t only and

$$f(y) = \begin{bmatrix} 1 \\ -\cosh y \\ \sin y \end{bmatrix} \quad \text{and} \quad h(y) = \begin{bmatrix} y \\ \sinh y \\ \cos y \end{bmatrix} \quad \text{for } \delta = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

By making use of (2.2) and (2.3), the field equations (1.1) and (1.2) for the metric (3.1), can be written as

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} = 0, \tag{3.2}$$

$$\frac{2\ddot{R}}{R}+\frac{\dot{R}^2+\delta}{R^2}-\frac{3S^2}{4R^4}=\frac{8\pi\lambda}{\phi}, \tag{3.3}$$

$$\frac{2\dot{R}\dot{S}}{RS}+\frac{\dot{R}^2+\delta}{R^2}-\frac{S^2}{4R^4}=\frac{8\pi\rho}{\phi}, \tag{3.4}$$

$$\ddot{\phi}+\dot{\phi}\left(\frac{\dot{S}}{S}+\frac{2\dot{R}}{R}\right)=-\frac{8\pi}{\phi}(\lambda+\rho). \tag{3.5}$$

2.3.1 Solutions of the Field Equations

We have four equations in five unknowns R, S, ϕ, λ, and ρ, so we make some simplifying assumptions to the metric potentials.

Let $R=t^m$ then from (3.2) we have

$$\frac{\ddot{S}}{S}+\frac{m}{t}\frac{\dot{S}}{S}+\frac{m(m-1)}{t^2}+\frac{S^2}{4t^{4m}}=0. \tag{3.6}$$

By integrating the above equation we obtain

$$S=t^{2m-1} \tag{3.7}$$

where m is an arbitrary constant and satisfies the equation $28m^2-32m+9=0$ which immediately gives $m=\frac{9}{14}$ and $m=\frac{1}{2}$.

Case 1: In case $m=\frac{9}{14}$, the general solution of the field equations (3.2–3.4) can be written as

$$R=t^{\frac{9}{14}}, \tag{3.8}$$

$$S=t^{\frac{2}{7}}, \tag{3.9}$$

$$8\pi\left(\frac{\lambda}{\phi}\right)=-\frac{39}{49}t^{-2}+\delta t^{-\frac{9}{7}}, \tag{3.10}$$

$$8\pi\left(\frac{\rho}{\phi}\right)=\frac{26}{49}t^{-2}+\delta t^{-\frac{9}{7}}. \tag{3.11}$$

It is observed that the above solution is formally similar to the solution obtained by Krori et al. [5, 6] in general relativity in the context of cosmic strings.

Here when

$$t\to 0,\qquad \frac{\rho_p}{|\lambda|}=\frac{5}{3} \tag{3.12}$$

and as

$$t\to\infty,\qquad \frac{\rho_p}{|\lambda|}\to\frac{65}{49}t^{-\frac{5}{7}}. \tag{3.13}$$

Equation (3.12) shows that in the early era the universe is dominated by massive strings, but according to (3.13), in the later phase the strings dominate over the particles.

From (3.5), we get

$$\ddot{\phi}+\frac{11}{7t}\dot{\phi}+\left(-\frac{13}{49t^2}+\frac{2\delta}{t^{-\frac{9}{7}}}\right)\frac{\eta}{3}\phi=0. \tag{3.14}$$

We now present the solutions of the field equations for the following cases:

Case 1.1: Bianchi type-II ($\delta = 0$). The solution of the field equations (3.4–3.6) are

$$R = t^{\frac{9}{14}}, \tag{3.15}$$

$$S = t^{\frac{2}{7}}, \tag{3.16}$$

$$8\pi\left(\frac{\lambda}{\phi}\right) = -\frac{39}{49}t^{-2}, \tag{3.17}$$

$$8\pi\left(\frac{\rho}{\phi}\right) = \frac{26}{49}t^{-2} \tag{3.18}$$

and the scalar field is obtained from the following equation

$$\ddot{\phi} + \frac{11}{7t}\dot{\phi} - \left(\frac{13}{49t^2}\right)\frac{\eta}{3}\phi = 0. \tag{3.19}$$

Integrating the above equation we obtain

$$\phi = c_1 t^{k_1} + c_2 t^{k_2} \tag{3.20}$$

where

$$k_1 = \frac{-2 + \sqrt{8 + \frac{26}{3}\eta}}{7}$$

and

$$k_2 = \frac{-2 - \sqrt{8 + \frac{26}{3}\eta}}{7}.$$

If $\eta > 0$, the scalar field is real and one of the nodes of the scalar field is growing while the other is decaying.

Spatial volume $\quad V^3 = \sqrt{-g} = t^{\frac{44}{14}} + t^{\frac{34}{14}}$,

Scalar expansion $\quad \theta = \dfrac{11}{7t}$,

Deceleration parameter $\quad q = -3\theta^2\left[\dot{\theta} + \dfrac{1}{3}\theta^2\right] = \dfrac{13310}{2401t^4}$,

Shear scalar $\quad \sigma^2 = \dfrac{25}{588t^2}$.

Again the model decelerates since $q > 0$. The model exhibits anisotropy as $\lim_{t\to\infty} \frac{\sigma}{\theta} =$ constant. Thus we have, in this case, a non-inflationary and anisotropic model.

Case 1.2: Bianchi type-VIII and IX models. For Bianchi type-VIII and IX models, the field variables R and S will be of the same form given by (3.8) and (3.9). But (3.14) fails to yield an exact solution for ϕ in case $\delta = \pm 1$.

4 Conclusions

Here we considered Bianchi type-I, II, VIII and IX space times in Barber's second self-creation theory of gravitation in the context of massive cosmic strings. It is observed that due to the presence of scalar field the power index 'm' of the metric coefficients has a range of values and the non-existence of cosmic strings in some situations. It is also observed that the solutions obtained in Bianchi type-II, VIII and IX are formally similar to the solutions obtained by Krori et al. [5, 6].

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